H-PROJECTIVELY EQUIVALENT KÄHLER MANIFOLDS AND GRAVITATIONAL INSTANTONS[1]

Dmitry A. Kalinin
Department of General
Relativity & Gravitation
Kazan State University
18 Kremlyovskaya Ul.
Kazan 420008 Russia
E-mail: dima@tcnti.kazan.ru, dmitry.kalinin@ksu.ru

1 Introduction

Kähler manifolds were introduced by P. A. Shirokov [17] and E. Kähler [13] in the first part of our century. Since that time they gained applications in a wide variety of fields both in mathematics and theoretical physics [1, 4, 8, 9, 11]. In particular, Kähler manifolds have been studied as models for finding the gravitational instantons which are of great importance for construction of quantum gravity [7, 16].

The goal of the present paper is to investigate four dimensional Kähler manifolds admitting H-projective mappings with special attention to Einstein-Kähler manifolds of this type which can be interpreted as field configurations of the gravitational instantons.

The notion of H-projective mappings was introduced by T. Otsuki and Y. Tashiro [15] as a generalization of projective mappings of Riemannian manifolds [2, 18]. At the present moment wide variety of Kähler manifolds *not admitting* H-projective mappings is known. At the same time, some general methods of finding H-projective mappings for given Kähler manifold were also developed [18, 19, 20]. However, the problem of finding Kähler metrics and connections admitting non-affine H-projective mappings is still unsolved ever in the case of lower dimensions. Some approaches to its solution was proposed earlier [3, 5, 10] by the author in co-laboration with Prof. A. V. Aminova.

[1] This work was partially supported by Russian Foundation for Basic Researches (grant No 96-0101031).

In the first part of the present paper four-dimensional Kähler manifolds admitting non-affine H-projective mappings are studied. It is proved that four-dimensional non-Einstein Kähler manifolds admitting H-projective mappings are generalized equidistant manifolds. Moreover, it is proved that four-dimensional generalized equidistant Kähler manifolds admit H-projective mappings in general case.

The second part of the paper is devoted to investigation of Einstein generalized equidistant Kähler manifolds which can be interpereted as field configurations of gravitational instantons. Explicit expression for the metrics of such manifolds is found for Ricci-flat case and the case of Einstein-Kähler manifold ($Ric = \kappa g$) with $\kappa \neq 0$.

The author is grateful to A. Aminova, K. Matsumoto and J. Mikeš for comments, useful discussions and suggestions. My special thanks are addressed to the referee for valuable remarks and corrections.

2 Differential geometry of Kähler manifolds

Let me start from reminding some relevant facts on differential geometry of Kähler manifolds [12, 18, 20].

An $2n$-dimensional smooth manifold M is called to be *almost complex* if the *almost complex structure* $J : TM \to TM,\ J^2 = -\mathrm{id}|_{TM}$ is defined in its tangent bundle. A tensor field N of the type (1,2) on M defined by the formula

$$N(X, Y) = 2([JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]),$$

for any vector fields X, Y is called *torsion* of J. If $N = 0$ then J is called to be *complex structure*. In this case (M, J) is called *complex manifold*.

Let (M, J) be a complex manifold. According to the Newlander-Nirenberg theorem [12], there exists an unique complex analytic manifold M^c coinciding with M as topological space and such that its complex analytic structure induces the complex structure J and the structure of differential manifold on M.

The tangent bundle TM^c is $\mathbf{C}$-linear isomorphic to the bundle TM with the structure of complex bundle induced by J so that there is a canonical $\mathbf{C}$-linear bundle isomorphism

$$TM \otimes_{\mathbf{R}} \mathbf{C} \cong TM^c \oplus \overline{TM^c} \tag{1}$$

where $TM \otimes_{\mathbf{R}} \mathbf{C}$ is the complexification of TM and the bar denotes the complex conjugation.

Let (U, z^α), $\alpha = 1, ..., n$ be a chart on M^c. If M is the complex manifold corresponding to M^c then we shall say that $(U, z^\alpha, \overline{z^\alpha})$, $\alpha = 1, ..., n$ (or simply $(U, z, \overline{z})$) is *complex chart* on M. Because of the isomorphism (1) vector fields $\partial_\alpha \equiv \partial/\partial z^\alpha, \partial_{\overline{\alpha}} \equiv \partial/\partial \overline{z^\alpha}$, $\alpha = 1, ..., n$ define a basis in $TM \otimes_{\mathbf{R}} \mathbf{C}$. Any real tensor field T on M can be uniquely extended to the smooth field of elements of

the "complexified" tensor algebra

$$\tilde{\mathbf{T}}_pM \equiv \bigoplus_{k_i=1}^{\infty}((T_pM^c)^{\otimes k_1} \otimes (\overline{T_pM^c})^{\otimes k_2} \otimes (T_p^*M^c)^{\otimes k_3} \otimes (\overline{T_p^*M^c})^{\otimes k_4}).$$

In the coordinate basis $(\partial_\alpha,\, \partial_{\overline{\alpha}})$, $\alpha = 1, ..., n$ this extension has the form

$$T = T^{i_1...i_r}_{j_1...j_s}\partial_{i_1} \otimes ... \otimes \partial_{i_r} \otimes dz^{j_1} \otimes ... \otimes dz^{j_s}, \quad T^{\overline{i_1}...\overline{i_r}}_{\overline{j_1}...\overline{j_s}} = \overline{T^{i_1...i_r}_{j_1...j_s}}.$$

Here the Latin indices varied from 1 to $2n$ run over the sets of bared $(\overline{\alpha}, \overline{\beta}, \overline{\gamma}, ...)$ and unbarred $(\alpha, \beta, \gamma, ...)$ Greek indices varied from 1 to n.

In particular, the complex structure J can be uniquely extended to $\mathbf{C}$-linear endomorphism in $TM \otimes_{\mathbf{R}} \mathbf{C}$. The action of complex structure on the elements of coordinate basis is defined by the formulae $J\partial_\alpha = i\partial_\alpha$, $J\partial_{\overline{\alpha}} = -i\partial_{\overline{\alpha}}$.

Let us call *holomorphic transformation* a coordinate transformation of the form $z'^\alpha = w^\alpha(z)$, $z'^{\overline{\alpha}} = \overline{w^\alpha(z)}$ where $w^\alpha(z)$ are complex analytic functions. Let X be a real vector field. If the Lie derivative L_XJ is equal to zero then X is called to be *holomorphic vector field*. The condition $L_XJ = 0$ in a complex chart $(U, z, \overline{z})$ yields $\partial_{\overline{\nu}}\xi^\mu = \partial_\nu\xi^{\overline{\mu}} = 0$, $\mu, \nu = 1, ..., n$. Using the holomorphic coordinate transformations, in a vicinity of a regular point any holomorphic vector field can be reduced to the form $X = \partial_1 + \partial_{\overline{1}}$.

A complex manifold (M, J) is called *Kähler manifold* if a pseudo Riemannian metric g can be defined on M satisfying [12, 20]

$$g(JX, JY) = g(X, Y), \qquad \nabla_X J = 0 \tag{2}$$

for any vector fields X, Y. Here ∇ is the Levi-Civita connection of the metric g. The 2-form

$$\Omega(X, Y) = g(JX, Y) \tag{3}$$

is called *fundamental* 2*-form* of Kähler manifold M. From Eqs. (2), (3) and the condition $J^2 = -\mathrm{id}|_{TM}$ it follows that Ω is closed: $d\Omega = 0$.

Let $(U, z, \overline{z})$ be a complex chart on (M, g, J). Then the components of the metric g, the complex structure J and the fundamental 2-form Ω in the coordinate basis are defined by the conditions

$$g_{\alpha\overline{\beta}} = \overline{g_{\overline{\alpha}\beta}}, \qquad g_{\alpha\beta} = g_{\overline{\alpha}\overline{\beta}} = 0, \tag{4}$$

$$J^\alpha_\beta = -J^{\overline{\alpha}}_{\overline{\beta}} = i\delta^\alpha_\beta, \qquad J^\alpha_{\overline{\beta}} = J^{\overline{\alpha}}_\beta = 0, \tag{5}$$

$$\Omega_{\alpha\overline{\beta}} = \overline{\Omega_{\overline{\alpha}\beta}} = ig_{\alpha\overline{\beta}}, \qquad \Omega_{\alpha\beta} = \Omega_{\overline{\alpha}\overline{\beta}} = 0 \tag{6}$$

while the condition $d\Omega = 0$ takes the form

$$\partial_\alpha g_{\beta\overline{\gamma}} = \partial_\beta g_{\alpha\overline{\gamma}}, \qquad \partial_{\overline{\alpha}} g_{\overline{\beta}\gamma} = \partial_{\overline{\beta}} g_{\overline{\alpha}\gamma}. \tag{7}$$

From here it follows that in U exists a real-valued function Φ obeying

$$g_{\alpha\overline{\beta}} = \partial_\alpha\partial_{\overline{\beta}}\Phi. \tag{8}$$

This function is called *Kähler potential* of the metric g. It is defined up to the *gauge transformations*

$$\Phi' = \Phi + f(z) + \overline{f(z)}. \tag{9}$$

where f is an appropriate holomorphic function. From (4)–(8) it follows that the only non-zero Christoffel symbols and Riemann tensor of the metric g are

$$\Gamma^{\alpha}_{\beta\nu} = \overline{\Gamma^{\overline{\alpha}}_{\overline{\beta}\overline{\nu}}} = g^{\alpha\overline{\mu}}\partial_\beta g_{\overline{\mu}\nu} \tag{10}$$

while non-zero components of Ricci tensor Ric are defined by the conditions

$$R^{\alpha}_{\beta\mu\overline{\nu}} = \overline{R^{\overline{\alpha}}_{\overline{\beta}\overline{\mu}\nu}} = -R^{\alpha}_{\beta\overline{\nu}\mu} = -\overline{R^{\overline{\alpha}}_{\overline{\beta}\nu\overline{\mu}}} = -\partial_{\overline{\nu}}\Gamma^{\alpha}_{\beta\mu}, \tag{11}$$

$$R_{\alpha\overline{\beta}} = \partial_\alpha\partial_{\overline{\beta}}\ln(\det(g_{\mu\overline{\nu}})), \qquad R_{\alpha\overline{\beta}} = \overline{R_{\overline{\alpha}\beta}}. \tag{12}$$

3 H-projective mappings of Kähler manifolds

A smooth curve $\gamma : t \mapsto x(t)$ on a Kähler manifold (M, g, J) of real dimension $2n > 2$ is called to be *H-planar curve* if its tangent vector $\chi \equiv dx/dt$ satisfies the equations

$$\nabla_\chi \chi = a(t)\chi + b(t)J(\chi)$$

where $a(t)$ and $b(t)$ are functions of the parameter t.

Let us consider two Kähler manifolds M, M' with metrics g, g' and complex structures J, J'. A diffeomorphism $f : M \to M'$ is called *H-projective mapping* if for any H-planar curve γ in M the curve $f \circ \gamma$ is H-planar curve in M'. If a pair of Kähler manifolds M and M' admit a non-affine H-projective mapping $f : M \to M'$ then we shall say that these two manifolds are *H-projectively equivalent.* Any non-affine H-projective mapping preserve the complex structure, i.e. $f_* \circ J = J' \circ f_*$ [19].

Necessary and sufficient condition for a diffeomorphism f to be H-projective mapping can be expressed by the equation [18, 20]

$$f_*^{-1}(\nabla'_{f_*X}(f_*Y)) - \nabla_X Y = p(Y)X + p(X)Y - p(JX)JY - p(JY)JX \tag{13}$$

where p is a closed 1-form ($dp = 0$) on M and ∇, ∇' are the covariant derivatives with respect to Levi-Civita connections of the metrics g, g'. If, in particular, $p = 0$, then H-projective mapping preserves the connection and is affine. We shall consider further only non-affine, i.e. proper H-projective mappings. The condition (13) is equivalent to the following equation

$$(\nabla\tilde{g})(X, Y, Z) = 2p(Z)\tilde{g}(X, Y) + p(X)\tilde{g}(Y, Z) + p(Y)\tilde{g}(X, Z) -$$
$$p(JX)\tilde{g}(Y, JZ) - p(JY)\tilde{g}(X, JZ)$$

where $\tilde{g} = f_* g'$ and X, Y, Z are vector fields on M. In a complex coordinates, setting $Y = \partial_\alpha$, $Z = \partial_{\overline{\beta}}$ and $W = \partial_\gamma$, we get with the help of (4) and (5)

$$g'_{\alpha\overline{\beta},\gamma} = 2g'_{\alpha\overline{\beta}}\psi_{,\gamma} + 2g'_{\gamma\overline{\beta}}\psi_{,\alpha}, \qquad g'_{\alpha\beta,\gamma} = g'_{\alpha\beta,\overline{\gamma}} = 0 \tag{14}$$

where g'_{ij} are components of the pullback f^*g', comma denotes the covariant derivation and $p = \psi_{,i}dx^i$. Note, that f^*g' is a Kähler metric on (M, J) because f preserves the complex structure. Hence, g'_{ij} obey the conditions similar to (4) and (7).

Using the *Sinyukov's transformation* [18]

$$a_{\alpha\overline{\beta}} = \overline{a_{\overline{\alpha}\beta}} = e^{2\psi} g'^{\lambda\overline{\mu}} g_{\alpha\overline{\mu}} g_{\lambda\overline{\beta}}, \qquad a_{\alpha\beta} = a_{\overline{\alpha}\overline{\beta}} = 0, \qquad g^{\alpha\overline{\beta}} = e^{-2\psi} a^{\alpha\overline{\beta}} \tag{15}$$

where $a^{\alpha\overline{\beta}} = a_{\mu\overline{\lambda}} g^{\alpha\overline{\lambda}} g^{\mu\overline{\beta}}$ and $(g'^{\alpha\overline{\beta}}) = (g'_{\alpha\overline{\beta}})^{-1}$, we can write (14) in the form

$$a_{\alpha\overline{\beta},\gamma} = \lambda_\alpha g_{\gamma\overline{\beta}}, \qquad a_{\alpha\overline{\beta},\overline{\gamma}} = \lambda_{\overline{\beta}} g_{\gamma\overline{\alpha}} \tag{16}$$

where

$$\lambda_\alpha = \overline{\lambda_{\overline{\alpha}}} = -2\psi_{,\nu} e^{2\psi} g'^{\nu\overline{\mu}} g_{\alpha\overline{\mu}}.$$

Transvecting (16) with $g^{\alpha\overline{\beta}}$, we find

$$\lambda_\gamma = \overline{\lambda_{\overline{\gamma}}} = \frac{1}{2}\partial_\gamma(g^{ij}a_{ij}) = \partial_\gamma\lambda, \qquad \lambda = a_{\alpha\overline{\beta}} g^{\alpha\overline{\beta}}. \tag{17}$$

From here it follows, that $\lambda_i dz^i = d\lambda$ for a real function λ.

The integrability conditions of (16) follows from the Ricci identity

$$2a_{kl,[ij]} = a_{sl} R^s_{kij} + a_{ks} R^s_{lij}. \tag{18}$$

For $(ijkl) = (\gamma\overline{\nu}\alpha\overline{\beta})$ and $(\gamma\nu\alpha\overline{\beta})$ using (11) we get

$$a_{\mu\overline{\beta}} R^\mu_{\alpha\gamma\overline{\nu}} + a_{\alpha\overline{\mu}} R^{\overline{\mu}}_{\overline{\beta}\gamma\overline{\nu}} = g_{\gamma\overline{\beta}} \lambda_{\alpha,\overline{\nu}} - g_{\alpha\overline{\nu}} \lambda_{\overline{\beta},\gamma}, \tag{19}$$

$$g_{\gamma\overline{\beta}} \lambda_{\alpha,\nu} - g_{\nu\overline{\beta}} \lambda_{\alpha,\gamma} = 0. \tag{20}$$

The remaining integrability conditions hold identitically or can be obtained from (19) and (20) by complex conjugation. Contracting (19) with $g^{\alpha\overline{\nu}}$ we find

$$-a_{\mu\overline{\beta}} R^\mu_\gamma + a_{\alpha\overline{\mu}} R^{\overline{\mu}}_{\overline{\beta}\gamma}{}^{\alpha} = g_{\gamma\overline{\beta}} g^{\alpha\overline{\nu}} \lambda_{\alpha,\overline{\nu}} - n\lambda_{\overline{\beta},\gamma}.$$

From here, using (17) and the identity $a_{\alpha\overline{\mu}} R^{\overline{\mu}}_{\overline{\beta}\gamma}{}^{\alpha} = a_{\alpha\overline{\mu}} R^{\alpha}_{\gamma\overline{\beta}}{}^{\overline{\mu}}$, it is easy to derive

$$a^\nu_\mu R^\mu_\gamma - a^\mu_\gamma R^\nu_\mu = 0. \tag{21}$$

Transvecting (20) with $g^{\gamma\overline{\beta}}$ we find $(n-1)\lambda_{\alpha,\nu} = 0$ which means that $\lambda_{\alpha,\nu} = 0$ and $\lambda^\alpha_{,\overline{\nu}} = 0$, or, because $\Gamma^\alpha_{\overline{\nu}i} = 0$,

$$\partial_{\overline{\nu}}\lambda^\alpha = 0, \qquad \partial_\nu\lambda^{\overline{\alpha}} = 0. \tag{22}$$

So, we come to the conclusion that $\Lambda = \lambda^i\partial_i$ is a holomorphic vector field. Using the holomorphic coordinate transformations Λ can be reduced to the form

$$\Lambda = \partial_1 + \partial_{\overline{1}}, \qquad \lambda^\alpha = \delta^\alpha_1, \qquad \lambda^{\overline{\alpha}} = \delta^\alpha_1. \tag{23}$$

Theorem 1 *Let f be a non-affine H-projective mapping of a Kähler manifold (M, g) on a Kähler manifold (M', g'). Let also $d\lambda = \lambda_\alpha dz^\alpha + \lambda_{\overline{\alpha}} dz^{\overline{\alpha}}$ be the exact 1-form defined by Eqs. (14) – (17). Then the real vector field $J\Lambda = i\lambda^\alpha \partial_\alpha - i\lambda^{\overline{\alpha}} \partial_{\overline{\alpha}}$ is infinitesimal isometry of M, i.e. the Killing equations hold: $L_{J\Lambda} g = 0$.*

Proof: Using Eqs. (10), (17) and (22) we find

$$-i\lambda_{\overline{\beta},\alpha} + i\lambda_{\alpha,\overline{\beta}} = -i\partial_\alpha \partial_{\overline{\beta}}(a_{\nu\overline{\mu}} g^{\nu\overline{\mu}}) + i\partial_{\overline{\beta}} \partial_\alpha (a_{\nu\overline{\mu}} g^{\nu\overline{\mu}}) = 0,$$

$$i\lambda_{\beta,\alpha} + i\lambda_{\alpha,\beta} = 0, \qquad -i\lambda_{\overline{\beta},\overline{\alpha}} - i\lambda_{\overline{\alpha},\overline{\beta}} = 0,$$

or $L_S g_{ij} \equiv S_{i,j} + S_{j,i} = 0$, where $S = J\Lambda$ and $S_i = g_{il} S^l$. Since f is non-affine mapping the vector field $\Lambda \neq 0$ and S is the infinitesimal isometry. *Q.E.D.*

If we make $\Lambda = \partial_1 + \partial_{\overline{1}}$ (see (23)), then the Killing equations take the form

$$(\partial_1 - \partial_{\overline{1}}) g_{\alpha\overline{\beta}} = 0. \tag{24}$$

Lemma 1 *If a Kähler manifold (M, g, J) admits an infinitesimal isometry X which is a holomorphic vector field, then the Kähler potential of g can be reduced to the form*

$$\Phi = \Phi(z^1 + z^{\overline{1}}, z^2, z^{\overline{2}}, ...). \tag{25}$$

Proof: Any holomorphic vector field can be locally reduced to the form $X = i(\partial_1 - \partial_{\overline{1}})$. Then the Killing equations take the form (24). From here using (8) we find

$$(\partial_1 - \partial_{\overline{1}}) g_{\alpha\overline{\beta}} = \partial_\alpha \partial_{\overline{\beta}} (\partial_1 - \partial_{\overline{1}}) \Phi = 0. \tag{26}$$

Hence, $(\partial_1 - \partial_{\overline{1}})\Phi = f(z) + h(\overline{z})$ where f is a holomorphic function and h is an antiholomorphic function. Similarly, because Φ is real we have $(\partial_1 - \partial_{\overline{1}})\Phi = -\overline{(\partial_1 - \partial_{\overline{1}})\Phi}$ and $h(\overline{z}) = -\overline{f(z)}$. Let us change the Kähler potential using the gauge transformations of the form

$$\Phi = \Phi' + \int f(z) dz^1 + \overline{\int f(z) dz^1} = \int f(z) dz^1 - \int h(\overline{z}) d\overline{z^1}.$$

Substituting this expression in (26), we obtain $(\partial_1 - \partial_{\overline{1}})\Phi' = 0$. From here we find $\Phi' = \Phi'(z^1 + z^{\overline{1}}, z^2, z^{\overline{2}}, ...)$. *Q. E. D.*

Let a Kähler manifold (M, g) admits a non-affine H-projective mapping. Then, according to Theorem 1 and Lemma 1 the Kähler potential can be reduced to the form (25). In this case (16) yields

$$a^\alpha_{\beta,\overline{\gamma}} = \partial_{\overline{\gamma}} a^\alpha_\beta = \delta^\alpha_1 g_{\beta\overline{\gamma}}, \qquad a^\alpha_\beta = g^{\alpha\overline{\sigma}} a_{\beta\overline{\sigma}}, \tag{27}$$

$$a^\alpha_{\beta,\gamma} = \lambda_{,\beta} \delta^\alpha_\gamma. \tag{28}$$

Integrating the first equation in (27), we find

$$a^\alpha_\beta = \delta^\alpha_1 \partial_\beta \Phi + h^\alpha_\beta \tag{29}$$

where h^α_β are holomorphic functions. From (17) $\lambda = \partial_1\Phi + h^\alpha_\alpha$. Since λ and $\partial_1\Phi$ are real we get

$$\lambda = \partial_1\Phi + n\rho, \qquad h^\alpha_\alpha = h^{\overline{\alpha}}_{\overline{\alpha}} \equiv n\rho = \mathrm{const} \tag{30}$$

where we have used the fact that a holomorphic function is real iff it is constant. Substituting (30) in (28), we get

$$a^\alpha_{\beta,\gamma} = g_{\beta\overline{1}}\delta^\alpha_\gamma. \tag{31}$$

In the next section we shall consider this equation for the case of a four-dimensional Kähler manifold.

4 Non-Einstein manifolds of dimension four

Let (M_4, g, J) be a non-Einstein ($Ric \neq \kappa g$) Kähler manifold of dimension $\dim_{\mathbf{R}} M_4 = 4$. Let M_4 admits a non-affine H-projective mapping on a Kähler manifold (M_4', g', J) and let a be the tensor field defined by (15). We introduce tensor field $b = L_{J\Lambda}a$ where $J\Lambda$ is the infinitesimal isometry defined by Theorem 1.

According to (29) and Lemma 1, in a complex coordinates where

$$\Lambda = \partial_1 + \partial_{\overline{1}}, \qquad J\Lambda = i(\partial_1 - \partial_{\overline{1}}) \tag{32}$$

we have

$$a^\alpha_\beta = \overline{a^{\overline{\alpha}}_{\overline{\beta}}} = \delta^\alpha_1\partial_\beta\Phi + f^\alpha_\beta(z^1, z^2) + \rho\delta^\alpha_\beta, \qquad f^\alpha_\alpha = f^{\overline{\alpha}}_{\overline{\alpha}} = 0, \tag{33}$$

$$\Phi = \Phi(z^1 + z^{\overline{1}}, z^2, z^{\overline{2}}) \tag{34}$$

where $f^\alpha_\beta \equiv h^\alpha_\beta - \rho\delta^\alpha_\beta$ are holomorphic functions and Φ is the Kähler potential. From here

$$b^\alpha_\beta = \overline{b^{\overline{\alpha}}_{\overline{\beta}}} = i(\partial_1 - \partial_{\overline{1}})a^\alpha_\beta = i\partial_1 f^\alpha_\beta, \quad b^{\overline{\alpha}}_\beta = b^\alpha_{\overline{\beta}} = 0, \quad b^\sigma_\sigma = b^{\overline{\sigma}}_{\overline{\sigma}} = 0, \tag{35}$$

Admissible coordinate and gauge transformations which don't change the form of vector field $\Lambda = \partial_1 + \partial_{\overline{1}}$ and the form (34) of Kähler potential are

$$z'^1 = z^1 + l(z^2), \qquad z'^2 = m(z^2), \tag{36}$$

$$\Phi' = \Phi + r\cdot(z^1 + \overline{z^1}) + u(z^2) + \overline{u(z^2)}, \qquad r \in \mathbf{R} \tag{37}$$

where l, m and u are holomorphic functions depending on z^2 only. Taking the Lie derivative along $J\Lambda$ from both parts of (21), we get

$$b^\nu_\mu R^\mu_\gamma - b^\mu_\gamma R^\nu_\mu = 0. \tag{38}$$

Using this formula it is possible to prove the following

Lemma 2 *If a non-Einstein four-dimensional Kähler manifold M_4 admits a non-affine H-projective mapping, then in a neighborhood of each point $p \in M_4$ exist complex coordinates in which the following relations hold*

$$a^\alpha_\beta = \delta^\alpha_1 \partial_\beta \Phi + f^\alpha_\beta(z^2) + \rho \delta^\alpha_\beta, \qquad (\partial_1 - \partial_{\overline{1}})\Phi = 0. \tag{39}$$

We have placed the proof, which is rather long and technical, in Appendix A so as not interrupt exposition.

Admissible coordinate and gauge transformations not changing (39) are defined by the formulas (36) and (37). Using these transformations one can reduce f^α_β to one of the following forms:
a) $f^\alpha_\beta = \delta^\alpha_2 \delta^1_\beta$ for $f^2_1 \neq 0$,
b) $f^\alpha_\beta = \mu \varepsilon_\beta \delta^\alpha_\beta$, $\varepsilon_\beta = (-1)^{\beta+1}$ for $f^2_1 = 0$.

If we admit the first possibility then we come to the contradiction with the assumption that M_4 is non-Einstein manifold (see proof in Appendix B).

In the second case we have

$$a^\alpha_\beta = \delta^\alpha_1 \partial_\beta \Phi + \mu \varepsilon_\beta \delta^\alpha_\beta + \rho \delta^\alpha_\beta, \qquad \varepsilon_\beta = (-1)^{\beta+1}, \qquad \mu = \mu(z^2)$$

and, from (21)

$$R^2_1 = 0, \qquad (\partial_1 \Phi + 2\mu) R^1_2 + \partial_2 \Phi (R^2_2 - R^1_1) = 0. \tag{40}$$

Using the symmetry and the reality of a we find

$$g_{1\overline{1}}(\mu - \overline{\mu}) = 0, \tag{41}$$

$$\begin{aligned} g_{2\overline{1}} \partial_{\overline{2}} \Phi - g_{1\overline{2}} \partial_2 \Phi &= g_{2\overline{2}}(\overline{\mu} - \mu), \\ g_{1\overline{1}} \partial_{\overline{2}} \Phi - g_{1\overline{2}} \partial_1 \Phi &= g_{1\overline{2}}(\overline{\mu} + \mu). \end{aligned} \tag{42}$$

If $g_{1\overline{1}} \neq 0$ then from (41) it follows that $\mu = \overline{\mu}$. In the case $g_{1\overline{1}} = g_{2\overline{2}} = 0$ we find from (42) that $\partial_1 \Phi = -(\mu + \overline{\mu})$. Similarly, by (7) and (8), we get

$$\partial_1 g_{1\overline{2}} = \partial_1 g_{2\overline{1}} = \partial_1 g_{2\overline{2}} = 0. \tag{43}$$

Therefore, from (12) we find $R^1_1 = R^2_2 = 0$ and from (40) it is easy to get $R^1_2(\mu - \overline{\mu}) = 0$. Hence $R^1_2 = 0$ for $\mu \neq \overline{\mu}$ and $R^i_j = 0$. So we find that M_4 is an Einstein manifold that contradicts to our initial assumption. Therefore, $\mu = \overline{\mu}$ =const.

Making the transformation $\rho \to \rho - \mu$, $\Phi \to \Phi + 2\mu(z^1 + \overline{z^1})$, we reduce a^α_β to the form

$$a^\alpha_\beta = \delta^\alpha_1 \partial_\beta \Phi + \rho \delta^\alpha_\beta. \tag{44}$$

The reality and the symmetry of the tensor a imply

$$\partial_2 \Phi = \varphi \partial_1 \Phi \tag{45}$$

where $\varphi = \varphi(z^2, \overline{z^2})$ is a complex function. Using (8) we get

$$g_{2\overline{1}} = \varphi g_{1\overline{1}}, \qquad g_{2\overline{2}} = \partial_{\overline{2}}\varphi\partial_1\Phi + \varphi\overline{\varphi}g_{1\overline{1}}.$$

Because $g_{2\overline{2}}$ is real, $\partial_{\overline{2}}\varphi\partial_1\Phi = \partial_2\overline{\varphi}\partial_{\overline{1}}\Phi$, and by Lemma 1

$$\partial_{\overline{2}}\varphi = \partial_2\overline{\varphi}. \tag{46}$$

This equation can be interpreted as the integrability condition of the system

$$\varphi = \partial_2 F, \qquad \overline{\varphi} = \partial_{\overline{2}} F \tag{47}$$

where F is a real function depending only on z^2 and $z^{\overline{2}}$. If the equation (46) holds, then (47) has a solution F. Substituting it in (45), we find

$$\partial_2\Phi = \partial_2 F\partial_1\Phi \tag{48}$$

where $\partial_2\partial_{\overline{2}}F \neq 0$, because otherwise $\det(g_{\alpha\overline{\beta}}) = 0$.

Because of (44), (45) and (48) the equation (16) holds identically. It means that any Kähler manifold whose Kähler potential in any complex chart obeys the equations (44), (45) and (48) admits non-affine H-projective mappings.

Now we find general solution of the equation (48) for an appropriate function real $F(z^2, \overline{z^2})$. Let $\tilde{F}(z^2, \overline{z^2})$ be a real function functionally independent from F. Rewriting (48) in the variables $u = F(z^2, \overline{z^2})$ and $v = \tilde{F}(z^2, \overline{z^2})$, we find

$$\partial_u\Phi + \frac{\partial_2\tilde{F}}{\partial_2 F}\partial_v\Phi = \partial_1\Phi. \tag{49}$$

From here, taking into account the reality of the functions F, u and v as well as the identity $(\partial_1 - \partial_{\overline{1}})\Phi = 0$, we find

$$\left(\frac{\partial_2\tilde{F}}{\partial_2 F} - \frac{\partial_{\overline{2}}\tilde{F}}{\partial_{\overline{2}}F}\right)\partial_v\Phi = 0.$$

Since F, $\tilde{F}$ are functionally independent $\partial_v\Phi = 0$ and by (49) $\partial_u\Phi - \partial_1\Phi = 0$. Therefore, the general solution of (48) has the form $\Phi = \mathcal{W}(z^1 + \overline{z^1} + F(z^2, \overline{z^2}))$ where $\mathcal{W}$ is an appropriate real function of one real variable.

From these relations the main result now follows

Theorem 2 *Let f be a non-affine H-projective mapping of a non-Einstein four-dimensional Kähler manifold (M_4, g, J) on a Kähler manifold (M'_4, g', J). Then in a neighborhood of each point $p \in M_4$ exist complex coordinates $(z^\alpha, z^{\overline{\alpha}})$, $\alpha = 1, \ldots, n$ in which Kähler potential Φ can be chosen in the form*

$$\Phi = \mathcal{W}(z^1 + \overline{z^1} + F(z^2, \overline{z^2})), \quad F = \overline{F}, \quad \partial_2\partial_{\overline{2}}F \neq 0, \quad \mathcal{W} \neq const \tag{50}$$

and the components of the metric g are defined by the formula

$$g_{\alpha\overline{\beta}} = \partial_\alpha\partial_{\overline{\beta}}\Phi. \tag{51}$$

In the same coordinate system the pullback f^*g' of the metric g' is defined by Eq. (15) where

$$a_{\alpha\overline{\beta}} = \overline{a_{\overline{\alpha}\beta}} = \partial_\alpha\Phi\partial_1\partial_{\overline{\beta}}\Phi + \rho\partial_\alpha\partial_{\overline{\beta}}\Phi, \quad a_{\alpha\beta} = a_{\overline{\alpha}\overline{\beta}} = 0, \quad \rho \in \mathbf{R}. \tag{52}$$

5 Generalized equidistant Kähler manifolds and gravitational instantons

A (pseudo)Riemannian manifold (M, g) is called *equidistant* [18] if it admits a covector field φ obeying the condition $(\nabla\varphi)(X, Y) = \rho g(X, Y)$ where ρ is a smooth function and X, Y are appropriate vector fields on M. If in (50) $\mathcal{W}(x) = \exp(x)$ then (51) defines the metrics of an equidistant Kähler manifolds. Conversely, it can be shown that the Kähler potential of any equidistant manifold can be reduced to the form [14, 19]

$$\Phi(z^1, \overline{z^1}, \ldots, z^n, \overline{z^n}) = \exp(z^1 + \overline{z^1} + F(z^2, \overline{z^2}, \ldots, z^n, \overline{z^n}))$$

for a real function F.

We now define a more general class of Kähler manifolds then those of equidistant manifolds. A Kähler manifold M is called to be *generalized equidistant* if in local complex coordinates its Kähler potential can be reduced to the form

$$\Phi = \mathcal{W}(z^1 + \overline{z^1} + F(z^2, \overline{z^2}, \ldots, z^n, \overline{z^n})), \qquad F = \overline{F}.$$

Let us consider a four-dimensional generalized equdistant Kähler manifold with the metric g given by (50) and the tensor field a defined by the equation (52). As it was shown in the previous section, Eq. (16) where $\lambda_\alpha = g_{\alpha\overline{1}}$, $\lambda_{\overline{\alpha}} = g_{\overline{\alpha}1}$ holds identically for such g and a. Therefore, we have the following

Theorem 3 *Any four-dimensional generalized equidistant Kähler manifold admits a non-affine H-projective mapping.*

J. Mikeš [14] have proved that equidistant Kähler manifolds admit non-affine H-projective mappings and Theorem 3 confirms this result for the case of four-dimensional manifolds.

It is well-known that Kähler manifolds of constant holomorphic sectional curvature admits H-projective mappings. It is easy to show that such manifolods are generalized equidistant with

$$\Phi = \ln(1 + \epsilon \exp(z^1 + \overline{z^1} + \ln(1 + \sum_2^n z^\alpha \overline{z^\alpha}))), \qquad \epsilon = \pm 1$$

for non-zero holomorphic sectional curvature and

$$\Phi = \exp(z^1 + \overline{z^1} + \ln(1 + \sum_2^n z^\alpha \overline{z^\alpha}))$$

in the flat case. In particular, $\mathbf{CP}^n$ and $\mathbf{C}^n$ are generalized equidistant manifolds.

It is possible also to construct the following class of the generalized equidistant manifolds. Let N be an algebraic submanifold in $\mathbf{C}^{n+1}$ defined by the equation

$$\mathcal{F}_N(z^2, \ldots, z^{n+1}) = 0$$

where the function $\mathcal{F}_N$ is a polynomial invariant with respect to the action of the group $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$ on $\mathbf{C}^{n+1}$ by multiplications. Then $M = N/\mathbf{C}^*$ is a $n-1$-dimensional algebraic submanifold in $\mathbf{CP}^n$. Taking in $\mathbf{CP}^n$ Kähler metric with the potential defined by the formula [12]

$$\Phi = \ln(z^1\overline{z^1} + z^2\overline{z^2} + \ldots + z^{n+1}\overline{z^{n+1}})$$

it is easy to see that M with induced metric is generalized equidistant manifold.

We now consider the Einstein generalized equidistant manifolds ($Ric = \kappa g$). In the case $\kappa = 0$ the manifolds are Ricci-flat. Hence, they possess hyper Kähler structure [7]. For any value of κ the Einstein-Kähler manifolds have various important applications in theoretical and mathematical physics [6, 7, 16]. In particular, such manifolds describe field configurations of gravitational instantons [16]. From the point of view of differential geometry the problem of finding four-dimensional Einstein-Kähler manifolds is also of great interest and leads to investigation of complex Monge-Ampère equation [7, 21].

Einstein-Kähler generalized equidistant manifolds are distinguished by the condition

$$\exp(-\kappa\Phi)\partial_1(\partial_1\Phi)^2\partial_2\partial_{\overline{2}}F = f(z)\overline{f(z)} \tag{53}$$

where $f(z)$ is an appropriate holomorphic function. By the use of coordinate transformations one can make $f(z)\overline{f(z)} = \mathrm{const}$ or $f(z)\overline{f(z)} = \mathrm{const}\exp(z^1+\overline{z^1})$. For simplicity we restrict our further consideration only to the first case. In the first case we have

$$\exp(-\kappa\mathcal{W})\mathcal{W}'\mathcal{W}''\partial_{2\overline{2}}F = \mathrm{const} \neq 0.$$

Because F depends on z^2, $\overline{z^2}$ only this equation can be rewritten as

$$\mathcal{W}'\mathcal{W}''\exp(-\kappa\mathcal{W}) = \mathrm{const}, \qquad \partial_{2\overline{2}}F = \mathrm{const} \neq 0 \tag{54}$$

whence

$$F(z^2, \overline{z^2}) = \gamma z^2\overline{z^2} + \tau(z^2 + \overline{z^2}) + \sigma \tag{55}$$

where γ, τ and σ are real constants.

For $\kappa = 0$ (Ricci-flat case) after integration of (54), we find

$$\mathcal{W} = A(x + B)^{3/2} + C, \qquad x = z^1 + \overline{z^1} + F(z^2, \overline{z^2}) \tag{56}$$

where A, B and C are some real constants. After substituting (55) in (56) and making the admissible coordinate change $z^1 \to z^1 + (\tau^2 - \sigma)/2$, $z^2 \to z^2 - \tau$, we obtain the following general expression for Kähler potential Φ

$$\Phi = A(z^1 + \overline{z^1} + \gamma z^2\overline{z^2})^{3/2}$$

where the constant C is omitted because it corresponds to the gauge transformations. From here it is easy to get the following expression for the metric in complex coordinates

$$ds^2 = \frac{3}{4}A(z^1 + \overline{z^1} + \gamma z^2\overline{z^2})^{-1/2}[dz^1d\overline{z^1} + \gamma z^2dz^1d\overline{z^2} + \gamma\overline{z^2}dz^2d\overline{z^1} +$$

$$2\gamma(z^1+\overline{z^1}+\frac{3\gamma}{2}z^2\overline{z^2})\,dz^2d\overline{z^2}]. \tag{57}$$

Introducing the real coordinates

$$x=\frac{z^1+\overline{z^1}}{2},\quad y=\frac{z^1-\overline{z^1}}{2i},\quad u=\frac{z^2+\overline{z^2}}{2},\quad v=\frac{z^2-\overline{z^2}}{2i},$$

we find from (57) the following form of the metric of four-dimensional Ricci-flat generalized equidistant manifolds

$$\begin{gathered} ds^2=\frac{3}{4}A(x+\gamma(u^2+v^2))^{-1/2}[dx^2+dy^2+ \\ 2\gamma(u\,dx\,du+u\,dy\,dv-v\,dx\,dv+v\,dy\,du) \\ 2\gamma(x+\frac{3\gamma}{2}(u^2+v^2))(du^2+dv^2)]. \end{gathered} \tag{58}$$

For the case $\kappa\neq 0$ we have from (54) and (55)

$$\mathcal{W}'\mathcal{W}''\exp(-\kappa\mathcal{W})=\mathrm{const}.$$

After first integration of this equation we get

$$\mathcal{W}'=-\frac{1}{\kappa}\,(B-Ae^{\kappa\mathcal{W}})^{1/3} \tag{59}$$

where A and B are some constants. From here it is easy to find the metric coefficients

$$g_{1\overline{1}}=\frac{-A}{3\kappa}\,e^{\kappa\mathcal{W}}(B-A\,e^{\kappa\mathcal{W}})^{-1/3}, \tag{60}$$

$$g_{1\overline{2}}=\frac{-A\gamma z^2}{3\kappa}\,e^{\kappa\mathcal{W}}(B-A\,e^{\kappa\mathcal{W}})^{-1/3}, \tag{61}$$

$$g_{2\overline{2}}=\frac{-A\gamma z^2\overline{z^2}}{3\kappa}\,e^{\kappa\mathcal{W}}\,(B-A\,e^{\kappa\mathcal{W}})^{-1/3}-\frac{\gamma}{\kappa}\,(B-A\,e^{\kappa\mathcal{W}})^{1/3} \tag{62}$$

where the function $\mathcal{W}$ has to be found from (59). Integrating (59) in the case $B\neq 0$ we get the following relation between the function $\mathcal{W}$ and its argument $x=z^1+\overline{z^1}+F(z^2,\overline{z^2})$ (here F is given by (55))

$$\begin{gathered} x+C=\frac{-3}{\kappa}\,(\frac{\arctan(\frac{B^{1/3}+2T}{\sqrt{3}B^{1/3}})}{\sqrt{3}B^{1/3}}- \\ \frac{\ln(-B^{1/3}+T)}{3B^{1/3}}+\frac{\ln(B^{2/3}+B^{1/3}T+T^2)}{6B^{1/3}}) \end{gathered} \tag{63}$$

where $T=(B-A\,e^{\kappa\mathcal{W}})^{1/3}$. In the case $B=0$ from (59) it is easy to find

$$\mathcal{W}=\frac{3}{\kappa}\,\ln(x+C) \tag{64}$$

where the additive constant $\frac{3}{\kappa}\,\ln(\frac{A^{1/3}}{3})$ is not written.

The equations (58), (60)–(64) define the metrics of Einstein generalized equidistant manifolds. The manifolds of this type can be interpreted as field configurations of gravitational instantons.

Appendix A

Here we provide the proof of Lemma 2.

It follows from (35) that b^α_β depend only on z. Holomorphic coordinate transformations don't change this result and can be used to make $b^1_2 = 0$.

Let $b^1_2 = 0$, consider the following three possibilities in Eq. (38).

1) Let $b^2_1 = 0$ and the tensor field b does not vanishes. Then either $b^1_1 = b^2_2 = 0$ that contradicts with the assumption about not vanishing of tensor b^i_j or, because M_4 is non-Einstein manifold, $R^1_2 = R^2_1 = 0$ and $R^1_1 \neq R^2_2$. In the last case it is possible to find such functions v_1 and v_2 that

$$b^\alpha_\beta = v_1 R^\alpha_\beta + v_2 \delta^\alpha_\beta. \tag{A.1}$$

2) If $b^2_1 \neq 0$ and $b^1_1 \neq 0$, then from (38) we have

$$R^1_2 = 0, \qquad \frac{R^1_1 - R^2_2}{2b^1_1} = \frac{R^2_1}{b^2_1},$$

hence, $b^1_1 - b^2_2 = v_1(R^1_1 - R^2_2)$, $b^2_1 = v_1 R^2_1 = 0$, $b^1_2 = v_1 R^1_2$ for some function v_1. By putting $b^\alpha_\beta = v_1 R^\alpha_\beta + \tilde{b}^\alpha_\beta$, we find $\tilde{b}^1_1 - \tilde{b}^2_2 = \tilde{b}^1_2 = \tilde{b}^2_1 = 0$ or $\tilde{b}^\alpha_\beta = v_2 \delta^\alpha_\beta$ where v_2 is some function in U.

3) At last, in the case $b^2_1 \neq 0$, $b^1_1 = b^2_2 = 0$ we get $R^1_2 = R^1_1 - R^2_2 = 0$. Hence, it is possible to find functions v_1, v_2 such that that (A.1) holds. We come to the conclusion that (A.1) describes all possible cases. In the similar way the relations

$$b^{\overline{\alpha}}_{\overline{\beta}} = \overline{v_1} R^{\overline{\alpha}}_{\overline{\beta}} + \overline{v_2} \delta^\alpha_\beta, \qquad b^{\overline{\alpha}}_\beta = v_1 R^{\overline{\alpha}}_\beta + v_2 \delta^{\overline{\alpha}}_\beta \equiv 0. \tag{A.2}$$

can be obtained. From here because of the reality and the symmetry of a, b and Ric it follows that v_1 and v_2 are real-valued functions, i.e. (A.1), (A.2) can be rewritten in the form of one tensor relation

$$b^i_j = v_1 R^i_j + v_2 \delta^i_j.$$

Since $b^i_i = 0$ we get $v_2 = -(v_1 R)/2n$, whence

$$b^\alpha_\beta = v_1 (R^\alpha_\beta - \frac{R}{2n} \delta^\alpha_\beta). \tag{A.3}$$

Because of (10), (31) and (35) we have $b^\alpha_{\beta,j} = 0$. Differentiating (A.3) and denoting $\mathcal{A} = \ln v_1$, one can find

$$\mathcal{A}_{,j} = -\frac{(R^\alpha_\beta - \delta^\alpha_\beta R/2n)_{,j}}{R^\alpha_\beta - \delta^\alpha_\beta R/2n}.$$

The right hand side of this relation doesn't depend on the variable $y^1 = \frac{1}{\sqrt{2}}(z^1 - z^{\overline{1}})$, hence, its left hand side shouldn't depend too. Because $\mathcal{A}$ is real we have

$$\mathcal{A} = \tilde{f}(z^1 + z^{\overline{1}}, z^2, z^{\overline{2}}) + i\tilde{\tau} \cdot (z^1 - z^{\overline{1}}), \qquad \tilde{\tau} \in \mathbf{R},$$

$$v_1 = \exp \mathcal{A} = f(z^1 + z^{\overline{1}}, z^2, z^{\overline{2}}) \exp(i\tau \cdot (z^1 - z^{\overline{1}})), \qquad \tau \in \mathbf{R}.$$

Then from (A.3) we get $b^\alpha_\beta = \exp(2i\tau z^1)\tilde{c}^\alpha_\beta(z^2)$. Taking into account (35)

$$f^\alpha_\beta = -i \int b^\alpha_\beta dz^1 = \exp(2i\tau z^1) c^\alpha_\beta(z^2) + d^\alpha_\beta(z^2), \qquad c^\alpha_\alpha = d^\alpha_\alpha = 0. \tag{A.4}$$

Using this equation we find from (21), (33)

$$\delta^\alpha_1 \partial_\mu \Phi R^\mu_\beta - \partial_\beta \Phi R^\alpha_1 + d^\alpha_\mu R^\mu_\beta - d^\mu_\beta R^\alpha_\mu = 0, \tag{A.5}$$

$$c^\alpha_\mu R^\mu_\beta - c^\mu_\beta R^\alpha_\mu = 0. \tag{A.6}$$

Let us consider the cases $c^2_1 \neq 0$ and $c^2_1 = 0$. Using the admissible transformations (36) in the first case one can reduce c^α_β to the form

$$c^\alpha_\beta = \delta^\alpha_1 \delta^2_\beta \phi + \delta^\alpha_2 \delta^1_\beta$$

where ϕ is a holomorphic function depending on z^2 only. Substituting this expression in (A.6), we find $\phi R^2_1 = R^1_2$, $R^1_1 = R^2_2$, whence, by (A.5) we have

$$(\partial_1 \Phi + 2d^1_1) R^1_2 = 0, \qquad (\partial_1 \Phi + 2d^1_1) R^2_1 = 0.$$

If $R^1_2 \neq 0$ or $R^2_1 \neq 0$, then $\partial_1 \Phi = -2d^1_1$ is a holomorphic function and $g_{1\overline{1}} = g_{1\overline{2}} = 0$, hence, the metric is degenerate. Therefore, $R^1_2 = R^2_1 = R^1_1 - R^2_2 = 0$ that contradicts with our assumption that M_4 is non-Einstein manifold.

So we have $c^2_1 = 0$. In this case by the use of the admissible coordinate transformations one can make

$$c^\alpha_\beta = \overline{c^{\overline{\alpha}}_{\overline{\beta}}} = c^1_1 \delta^\alpha_\beta \varepsilon_\beta, \qquad \varepsilon_\beta = (-1)^{\beta+1}, \tag{A.7}$$

$$d^\alpha_\beta = \overline{d^{\overline{\alpha}}_{\overline{\beta}}} = d^1_1 \delta^\alpha_\beta \varepsilon_\beta + \gamma(z^2) \delta^\alpha_1 \delta^2_\beta + \zeta \delta^\alpha_2 \delta^1_\beta, \quad \zeta = 0, 1.$$

After the gauge transformation $\Phi \to \Phi + \int \gamma(z^2) dz^2 + \overline{\int \gamma(z^2) dz^2}$ we find taking into account (33) and (A.4)

$$d^\alpha_\beta = \overline{d^{\overline{\alpha}}_{\overline{\beta}}} = d^1_1 \delta^\alpha_\beta \varepsilon_\beta + \zeta \delta^\alpha_2 \delta^1_\beta. \tag{A.8}$$

Substituting (A.7) into (A.6), we get $c^1_1 R^1_2 = 0$, $c^1_1 R^2_1 = 0$, whence, $c^1_1 = 0$ or $R^1_2 = R^2_1 = 0$. In the last case from (A.5) and (A.8) follows $\partial_2 \Phi (R^2_2 - R^1_1) = 0$. Since $\partial_2 \Phi \neq 0$, we find $R^2_2 - R^1_1 = 0$. So M_4 is an Einstein manifold again, therefore, $c^1_1 = 0$ and $f^\alpha_\beta = d^\alpha_\beta(z^2)$. Substituting this result into (33) prove the Lemma 2.

Appendix B

Here we prove that the condition $f^{\alpha}_{\beta} = \delta^{\alpha}_{2}\delta^{1}_{\beta}$ contradicts with the assumption that the considered Kähler manifold is non-Einstein.

From (21) and (39) we find

$$R^1_2 = \partial_2\Phi R^2_1, \quad R^1_2\partial_1\Phi + \partial_2\Phi(R^2_2 - R^1_1) = 0. \tag{B.1}$$

Writing down the symmetry conditions of a we obtain with the help of (39) the next formulae

$$g_{2\overline{1}} = g_{1\overline{2}}, \qquad g_{1\overline{1}}\partial_2\Phi = g_{2\overline{1}}\partial_{\overline{1}}\Phi + g_{2\overline{2}},$$
$$g_{1\overline{2}}\partial_1\Phi + g_{2\overline{2}} = g_{1\overline{1}}\partial_2\Phi,$$
$$g_{1\overline{2}}\partial_2\Phi - g_{2\overline{1}}\partial_{\overline{2}}\Phi \equiv g_{1\overline{2}}(\partial_2\Phi - \partial_{\overline{2}}\Phi) = 0.$$

From the last equation it follows that either $\partial_2\Phi = \partial_{\overline{2}}\Phi$ or $g_{1\overline{2}} = g_{2\overline{1}} = 0$.

Let us first take $g_{2\overline{1}} = g_{1\overline{2}} = 0$, then from (7) the equality $\partial_1 g_{2\overline{2}} = \partial_2 g_{1\overline{1}} = 0$ follows, hence $\partial_1\partial_{\overline{2}}\det(g_{\alpha\overline{\beta}}) = 0$ and, because of (12) we find $R_{1\overline{2}} = R_{2\overline{1}} = 0$, therefore, $R^2_1 = R^1_2 = 0$. Since $\partial_2\Phi \neq 0$, from (B.1) we get $R^1_1 - R^2_2 = 0$, which means that M_4 is Einstein manifold. We came to contradiction with our initial assumption. Hence, in addition to the formula $\partial_1\Phi = \partial_{\overline{1}}\Phi$ we have $\partial_2\Phi = \partial_{\overline{2}}\Phi$. From here using Eqs. (8) – (12), it is possible to deduce that all components of the metric tensor, Christoffel symbols and curvature tensor are real. Then (21) can be written as

$$R_{\alpha\overline{\sigma}}a^{\overline{\sigma}}_{\overline{\beta}} - R_{\sigma\overline{\beta}}a^{\sigma}_{\alpha} = 0.$$

From here, putting $\alpha, \beta = 1, 2$ and using the identities $\partial_2\Phi \neq 0$, $a^{\alpha}_{\beta} = a^{\overline{\alpha}}_{\overline{\beta}}$ and $R_{\alpha\overline{\beta}} = R_{\overline{\alpha}\beta}$, we find $R_{\alpha\overline{\beta}} = 0$, hence, $Ric = 0$ that contradicts with the assumption that M_4 is non-Einstein. *Q.E.D.*

References

[1] L. Alvarez-Gaume and D. Z. Freedman, Kähler geometry and the renormalization of supersymmetric sigma-models, *Phys. Rev.*, **D22** (1986), 846-870.

[2] A. V. Aminova, Pseudo Riemannian manifolds with common geodesics, *Russ. Math. Surveys*, **48** (1993), 107-159.

[3] A. V. Aminova and D. A. Kalinin, H-projectively equivalent four-dimensional Riemannian connections, *Russian Math. (Izv VUZ),* **38** (1994), 11-21.

[4] A. V. Aminova and D. A. Kalinin, Quantization of Kähler manifolds admitting H-projective mappings. *Tensor*, **56** (1995), 1-13.

[5] A. V. Aminova and D. A. Kalinin, H-projective mappings of four-dimensional Kähler manifolds, to appear in *Russian Math. (Izv VUZ).*

[6] A. V. Aminova, S. V. Zuev and D. A. Kalinin, On the quaternionic structure of target space and instanton solutions in gravity theory, *Abstr. of Contr. Papers. Int. Conf. GR14, Florence*, 1995, A88.

[7] A. L. Besse, *Einstein manifolds*, V.II, Springer, Berlin, 1987.

[8] M. J. Bowick and S. G. Rajeev, String theory as a Kähler geometry of loop space, *Phys. Rev. Lett.*, **58** (1987), 535-544.

[9] E. J. Flaherty, *Hermitian and Kähler geometry in relativity*, Springer, Berlin, 1976.

[10] D. A. Kalinin, Geometry of quantum systems with Kähler structure, *PhD Thesis,* Kazan State University, 1996.

[11] D. A. Kalinin, Trajectories of charged particles in Kähler magnetic fields, *Rept. Math. Phys.*, **39** (1997), 299-309.

[12] S. Kobayashi and K. Nomizu, *Foundation of differential geometry*, V.II, Intersci. Publ., N.Y., 1969.

[13] E. Kähler, Über eine bemerkenswerte Hermitishe Metric, *Abh. Math. Semin. Hamburg*, **9** (1933), 173-180.

[14] J. Mikeš, On equidistant Kähler spaces, *Matem. Zametki*, **38** (1985), 627-633.

[15] T. Otsuki and Y. Tashiro, On curves in Kahlerian spaces, *Math. J. Okayama Univ.*, **4** (1954), 57-78.

[16] M. Perry, Gravitational instantons, *Seminar on Differential Geometry*, Ed. S.-T. Yau, Princeton Univ. Press. Princeton, N.J., 1982, 603-630.

[17] P. A. Shirokov, Constant fields of vectors and tensors of 2-nd order in Riemannian spaces. *Izvestiya fiz.-mat. ob-va Kazan. gos. univ-ta*, **25** (1925), 256-280.

[18] N. S. Sinyukov, *Geodesic mappings of Riemannian spaces*, Nauka, Moscow, 1979.

[19] N. S. Sinyukov, I. N. Kurbatova and J. Mikeš, *Holomophic-projective mappings of Kähler spaces*, Odessa State Univ. Publ., Odessa, 1985.

[20] K. Yano, *Differential geometry of complex and almost complex spaces*, Pergamon Press, Oxford, 1965.

[21] S.-T. Yau, Survey on partial differential equations in differential geometry, *Seminar on Differential Geometry*, Ed. S.-T. Yau, Princeton Univ. Press. Princeton, N.J., 1982, 3-71.